

Contents lists available at [SciVerse ScienceDirect](http://SciVerse.ScienceDirect.com)

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Maximal subalgebras of the octonions

Stephen M. Gagola III*

Case Western Reserve University, Department of Mathematics, Cleveland, OH 44106-7058, USA

ARTICLE INFO

Article history:

Received 3 February 2008

Received in revised form 21 February 2012

Available online 9 July 2012

Communicated by M. Sapir

MSC:

Primary: 17A75

Secondary: 17A35; 17D05

ABSTRACT

Here we categorize all of the maximal subalgebras of the octonion algebras.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

In his paper [1], Michel Racine classified the maximal subalgebras of the octonions in Theorem 5. Unfortunately, this theorem holds only when the field is not an imperfect field of characteristic 2. The aim of this paper is to correct the defect.

Let F be a field. A unital algebra over a field F is a *composition algebra* if it has a norm that is multiplicative, is non-degenerate, and is nonzero for $F \neq 1$. A composition algebra is called a *division algebra* if it contains no zero-divisors. Otherwise it is said to be *split*.

Theorem 1. *If C is an octonion algebra over a field F and A is a maximal subalgebra of C then one of the following holds:*

1. C is split and A is the stabilizer of some hexagon line S , namely $A = S^\perp$;
2. $A = Q$ for some nonsplit quaternion subalgebra, Q ;
3. A is a four-dimensional, totally isotropic, totally nonsingular, commutative, associative subalgebra of C and F is a nonperfect field of characteristic 2.

Here, a hexagon line is a totally singular 2-space that is contained in 1^\perp such that the multiplication is zero. A version of this theorem was proven by M.L. Racine [1]. In Racine's result case 3 does not occur; however, such examples do exist. See [Example 2](#) below.

2. Main result

In his paper [1], Michel Racine misses just one case in his classification of the maximal subalgebras of the Cayley algebras.

Proof of Theorem 1. To correct the original proof of the foregoing result, it is enough to observe that if A is a maximal subalgebra that is simple, then it may happen that A is a purely inseparable field extension of F with $\text{char}(F) = 2$. In this case $\dim(A) = 4$, otherwise, by Artin's Theorem, the subalgebra generated by $A \cup \{a\}$, for any $a \in C \setminus A$, would be associative and therefore a proper subalgebra contradicting the maximality of A . \square

* Correspondence to: Department of Algebra, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic.
E-mail address: gagola@karlin.mff.cuni.cz.

Below is an example of an octonion algebra that contains a maximal subalgebra that is both totally isotropic and totally nonsingular, and hence does not fall under either case 1 or case 2 of [Theorem 1](#).

Example 2. Consider Zorn's vector matrix algebra C over a field F . If F is a nonperfect field of characteristic 2 and the elements $1, \alpha, \beta, \alpha\beta \in F$ are linearly independent over the subfield F^2 then a nice example of a maximal totally isotropic subalgebra of the split octonions, C , is

$$A = \left\langle \begin{pmatrix} 0 & (\alpha, 0, 0) \\ (1, 0, 0) & 0 \end{pmatrix}, \begin{pmatrix} 0 & (0, \beta, 0) \\ (0, 1, 0) & 0 \end{pmatrix} \right\rangle.$$

Here A is a commutative, associative, subalgebra with dimension 4. Since $1, \alpha, \beta$, and $\alpha\beta$ are linearly independent over the field F^2 of characteristic 2, A is both totally isotropic and totally nonsingular. Therefore, it is contained in neither a quaternion subalgebra nor the stabilizer of a hexagon line. So this is an example of a subalgebra that falls under case 3 of [Theorem 1](#) and not under cases 1 or 2.

Acknowledgments

The author is grateful for the referee's careful reading and useful comments.

References

- [1] M.L. Racine, Maximal subalgebras, J. Algebra 30 (1974) 155–180.